Quantization of Pre-Quasi-Symplectic Groupoids and their Hamiltonian Spaces

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Dedicated to Alan Weinstein on the occasion of his 60th birthday

Abstract

We study the prequantization of pre-quasi-symplectic groupoids and their Hamiltonian spaces using S^1-gerbes. We give a geometric description of the integrality condition. As an application, we study the prequantization of the quasi-Hamiltonian G-spaces of Alekseev–Malkin–Meinrenken.

Contents

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1 Introduction

Quasi-symplectic groupoids are natural generalizations of symplectic groupoids [7, 22]. The main motivation of [22] in studying quasi-symplectic groupoids was to introduce a single, unified momentum map theory in which ordinary Hamiltonian G-spaces, Lu's momentum maps of Poisson group actions, and the group-valued momentum maps of Alekseev–Malkin–Meinrenken can be understood under a uniform framework. An important feature of this unified theory is that it allows one to understand the diverse theories in such a way that techniques in one can be applied to the others.

It turns out that much of the theory of Hamiltonian spaces of a symplectic groupoid can be generalized to quasi-symplectic groupoids. In particular, one can perform reduction and prove that $J^{-1}(\mathcal{O})/\Gamma$ is a symplectic manifold, where $\mathcal{O} \subset M$ is an orbit of the groupoid $\Gamma \rightrightarrows M$. More generally, one can introduce the classical intertwiner space $\overline{X_2} \times_\Gamma X_1$ between two Hamiltonian Γ-spaces X_1 and X_2, generalizing the notion studied by Guillemin-Sternberg [10] for ordinary Hamiltonian G-spaces. One shows that this is a symplectic manifold (whenever it is a smooth manifold). In particular, when Γ is the AMM quasi-symplectic groupoid [6, 22], this reduced space describes the symplectic structure on the moduli space of flat connections on a surface [3].

As is the case for symplectic groupoids [20], one can introduce Morita equivalence for quasi-symplectic groupoids. In particular, it has been proven [22] that (i) Morita equivalent quasi-symplectic groupoids give rise to equivalent momentum map theories, in the sense that there is a bijection between their Hamiltonian spaces; (ii) the classical intertwiner space $\overline{X_2} \times_\Gamma X_1$ is independent of the Morita equivalence class of Γ. This Morita invariance principle accounts for various well-known results concerning the equivalence of momentum maps, including the Alekseev–Ginzburg–Weinstein linearization theorem [1, 9] and the Alekseev–Malkin–Meinrenken equivalence theorem for group-valued momentum maps [3] (see [22] for details).

One important feature of Hamiltonian G-spaces is the Guillemin–Sternberg theorem which states that "$[Q, R] = 0$": quantization commutes with reduction [10, 13]. One expects that "$[Q, R] = 0$" should be a general guiding principle for all momentum map theories. To carry out such a quantization program, the first important step is the construction of prequantum line bundles. In this paper, we study the prequantization of Hamiltonian spaces for quasi-symplectic groupoids. Our method uses the theory of S^1-bundles and S^1-gerbes over a groupoid along with their characteristic classes, as developed in [4, 5]. Roughly, our construction can be described as follows. A prequantization of a quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$ is an S^1-central

extension $R \to \Gamma$ of the groupoid $\Gamma \rightrightarrows M$ (or an S^1-gerbe over the groupoid) equipped with a pseudo-connection having $\omega + \Omega$ as pseudo-curvature. Such a prequantization exists if and only if $\omega + \Omega$ is a de Rham integral 3-cocycle and Ω is exact (assuming that Γ is a proper groupoid). A prequantization of a Hamiltonian space is then an S^1-bundle L over $R \rightrightarrows M$ together with a compatible pseudo-connection, where the R-action on L is S^1-equivariant. A prequantization of the symplectic intertwiner space $\overline{X_2} \times_\Gamma X_1$ can be constructed using these data.

Indeed one can show that $R \backslash (\overline{L_1} \times_M L_2)$ is a prequantization of the symplectic intertwiner space $\overline{X_2} \times_\Gamma X_1$, and the natural 1-form on $\overline{L_1} \times_M L_2$ induced by the connection forms on L_1 and L_2 descends to a prequantization connection on the quotient space $R \backslash (\overline{L_1} \times_M L_2)$. When Ω is not exact, one must pass to a Morita equivalent quasi-symplectic groupoid first. Then the Morita invariance principle guarantees that the resulting quantization does not depend on the particular choice of Morita equivalent quasi-symplectic groupoid. As a special case, when Γ is the AMM quasi-symplectic groupoid, our construction yields the prequantization of quasi-Hamiltonian G-spaces of Alekseev–Malkin–Meinrenken and their symplectic reductions, and our quantization condition coincides with that of Alekseev–Meinrenken [2].

Quantization of Hamiltonian spaces for symplectic groupoids was studied in [21]. Note that in the usual Hamiltonian case, since the symplectic 2-form defines a zero class in the third cohomology group of the groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, which is the equivariant cohomology $H^3_G(\mathfrak{g}^*)$, gerbes do not appear explicitly. However, for a general quasi-symplectic groupoid (for instance the AMM quasi-symplectic groupoid), since the 3-cocycle $\omega + \Omega$ may define a nontrivial class, gerbes are inevitable in the construction. Also note that no nondegeneracy condition is needed in the quantization construction, so we drop this assumption in the present paper to assure full generality.

This paper is organized as follows. In Section 2, we review some basic results concerning pre-quasi-symplectic groupoids and their Hamiltonian spaces. In Section 3, we gather some important results on S^1-bundles and S^1-central extensions. We give a simple formula for the index of an S^1-bundle over a central extension in terms of the Chern class. In Section 4, we introduce pre-quantizations of pre-quasi-symplectic groupoids and discuss compatible prequantizations of their Hamiltonian spaces. Section 5 is devoted to the description of a geometric integrality condition of pre-Hamiltonian Γ-spaces. The application to quasi-Hamiltonian G-spaces is discussed.

Unless specified, by a groupoid in this paper, we always mean a Lie groupoid whose orbit space is connected. A remark is in order concerning the terminology. In [7], quasi-symplectic groupoids are called presymplectic groupoids, where some "non-degeneracy" condition is assumed. Here we choose to use the "quasi" part of the terminology to refer to the presence of a 3-form and to use "pre-" to men that "non-degeneracy" is flexible.

Note that it would be interesting to investigate what notion of polarization would be relevant for the next step of this quantization scheme.

Prequantization of symplectic groupoids was first studied by Alan Weinstein and the second author in [19], when the second author was his PhD student. In the same paper, S^1-central extensions of Lie groupoids were also systematically investigated for the first time. Undoubtedly, Alan Weinstein's work and insights have had a tremendous impact on the development of this subject in the past two decades. It is our great pleasure to dedicate this paper to him.

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2 Pre-Hamiltonian Γ-spaces and classical intertwiner spaces

2.1 Pre-quasi-symplectic groupoids and their pre-Hamiltonian spaces

First, let us recall the definition of the de Rham double complex of a Lie groupoid. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Define for all $p \geq 0$

$$\Gamma_p = \underbrace{\Gamma \times_M \ldots \times_M \Gamma}_{p \text{ times}},$$

i.e., Γ_p is the manifold of composable sequences of p arrows in the groupoid $\Gamma \rightrightarrows M$ (and $\Gamma_0 = M$). We have $p+1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ (each leaving out one of the $p+1$ objects involved in a sequence of composable arrows), giving rise to a diagram

$$\ldots \Gamma_2 \mathrel{\substack{\longrightarrow\\[-0.6ex]\longrightarrow\\[-0.6ex]\longrightarrow}} \Gamma_1 \rightrightarrows \Gamma_0 \,. \tag{1}$$

Consider the double complex $\Omega^\bullet(\Gamma_\bullet)$:

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \\
\Omega^1(\Gamma_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^1(\Gamma_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^1(\Gamma_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots \\
\Omega^0(\Gamma_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^0(\Gamma_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^0(\Gamma_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots
\end{tikzcd} \tag{2}$$

Its boundary maps are $d : \Omega^k(\Gamma_p) \to \Omega^{k+1}(\Gamma_p)$, the usual exterior derivative of differentiable forms and $\partial : \Omega^k(\Gamma_p) \to \Omega^k(\Gamma_{p+1})$, the alternating sum of the pull-back maps of (1). We denote the total differential by $\delta = (-1)^p d + \partial$. The cohomology groups of the total complex $C^\bullet_{dR}(\Gamma_\bullet)$

$$H^k_{dR}(\Gamma_\bullet) = H^k\big(\Omega^\bullet(\Gamma_\bullet)\big)$$

are called the *de Rham cohomology* groups of $\Gamma \rightrightarrows M$.

Definition 2.1 A pre-quasi-symplectic groupoid is a Lie groupoid $\Gamma \rightrightarrows M$ equipped with a 2-form $\omega \in \Omega^2(\Gamma)$ and a 3-form $\Omega \in \Omega^3(M)$ such that

$$d\Omega = 0, \quad d\omega = \partial\Omega \ \text{ and } \ \partial\omega = 0. \tag{3}$$

In other words, $\omega + \Omega$ is a 3-cocycle of the total de Rham complex of the groupoid $\Gamma \rightrightarrows M$.

A pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$ is said to be *exact* if Ω is an exact 3-form on M.

A quasi-symplectic groupoid is a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega+\Omega)$ where ω satisfies certain non-degenerate condition [7, 22]. Quasi-symplectic groupoids are natural generalization of symplectic groupoids, whose momentum map theory unifies various momentum map theories, including the ordinary Hamiltonian G-spaces, Lu's momentum maps of Poisson group actions, and group valued momentum maps of Alekseev–Malkin–Meinrenken.

Definition 2.2 Given a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$, a pre-Hamiltonian Γ-space is a (left) Γ-space $J : X \to M$ (*i.e.*, Γ acts on X from the left) with a compatible 2-form $\omega_X \in \Omega^2(X)$ such that:

1. $d\omega_X = J^*\Omega$;
2. the graph of the action $\Lambda = \{(r, x, rx) | t(r) = J(x)\} \subset \Gamma \times X \times \overline{X}$ (where $\overline{X}$ is the manifold X endowed with the form $-\omega_X$) is isotropic with respect to the 2-form $(\omega, \omega_X, -\omega_X)$.

To illustrate the intrinsic meaning of the above compatibility condition, let us elaborate it in terms of groupoids. Let $\Gamma \times_M X \rightrightarrows X$ be the transformation groupoid corresponding to the Γ-action, and, by abuse of notation, $J : \Gamma \times_M X \to \Gamma$ the natural projection. It is simple to see that

$$\begin{array}{ccc} \Gamma \times_M X & \xrightarrow{J} & \Gamma \\ \downdownarrows & & \downdownarrows \\ X & \xrightarrow{J} & M \end{array} \tag{4}$$

is a Lie groupoid homomorphism. Therefore it induces a map, *i.e.*, the pull-back map, on the level of the de Rham complex:

$$J^* : \quad \Omega^\bullet(\Gamma_\bullet) \to \Omega^\bullet((\Gamma \times_M X)_\bullet).$$

Proposition 2.3 *[22] Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be a pre-quasi-symplectic groupoid and $J : X \to M$ a left Γ-space. A 2-form $\omega_X \in \Omega^2(X)$ is compatible with the action if and only if*

$$J^*(\omega + \Omega) = \delta\omega_X. \tag{5}$$

2.2 Classical intertwiner spaces

Consider a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$, and pre-Hamiltonian Γ-spaces $(X_1 \overset{J_1}{\to} M, \omega_1)$, and $(X_2 \overset{J_2}{\to} M, \omega_2)$. Assume that $\Gamma \backslash (\overline{X_2} \times_M X_1)$ is a smooth manifold, and denote by

$$p : \overline{X_2} \times_M X_1 \to \Gamma \backslash (\overline{X_2} \times_M X_1)$$

the natural projection. Note that $i^*(-\omega_2, \omega_1)$, where $i : \overline{X_2} \times_M X_1 \to X_2 \times X_1$ is the natural embedding, is a closed 2-form on $\overline{X_2} \times_M X_1$.

Proposition 2.4 *The 2-form $i^*(-\omega_2, \omega_1)$ descends to a closed 2-form on $\Gamma \backslash (\overline{X_2} \times_M X_1)$. Therefore $\Gamma \backslash (\overline{X_2} \times_M X_1)$ is a presymplectic manifold.*

To prove this proposition, we need a technical lemma.

Lemma 2.5 *[11] Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $X \to M$ a left Γ-space. Assume that $\Gamma \backslash X$ is a smooth manifold. A differential form $\omega \in \Omega^*(X)$ descends to a differential form on the quotient $\Gamma \backslash X$ if and only if $\partial\omega = 0$, where ∂ is with respect to the transformation groupoid $\Gamma \times_M X \rightrightarrows X$.*

Proof of Proposition 2.4. Note that the manifold $X_2 \times_M X_1$ with the momentum map $J : X_2 \times_M X_1 \to M$, $J(x_2, x_1) = J_1(x_1) = J_2(x_2)$, is naturally a Γ-space, where $\Gamma \rightrightarrows M$ acts on $X_2 \times_M X_1$ diagonally. Then

$$\begin{aligned} \partial[i^*(-\omega_2, \omega_1)] &= i^*(-\partial\omega_2, \partial\omega_1) \\ &= i^*(-J_2^*\omega, J_1^*\omega) \\ &= \big((J_2 \times J_1)\circ i)^*(-\omega, \omega\big) \\ &= 0, \end{aligned}$$

where J_1, J_2 and i are respectively the groupoid homomorphisms:

$$\begin{array}{ccc} \Gamma \times_M X_k & \xrightarrow{J_k} & \Gamma \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_k & \xrightarrow{J_k} & M \end{array} \quad k = 1, 2. \tag{6}$$

and

$$\begin{array}{ccc} \Gamma \times_M (X_1 \times_M X_2) & \xrightarrow{i} & (\Gamma \times_M X_1) \times (\Gamma \times_M X_2) \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_1 \times_M X_2 & \xrightarrow{i} & X_1 \times X_2 \end{array} , \tag{7}$$

and $\partial[i^*(-\omega_2, \omega_1)]$ and $\partial\omega_k$, $k = 1,\ 2$, are with respect to the groupoids on the left-hand side of Eqs. (7) and (6), respectively.

The conclusion thus follows from Lemma 2.5. $\square$

The presymplectic manifold $\Gamma\backslash(\overline{X_2} \times_M X_1)$ is called the *classical intertwiner space*, and is denoted by $\overline{X_2} \times_\Gamma X_1$ for simplicity. In particular, if $(\Gamma \rightrightarrows M, \omega + \Omega)$ is a quasi-symplectic groupoid, and $(X_1 \overset{J_1}{\to} M, \omega_1)$ and $(X_2 \overset{J_2}{\to} M, \omega_2)$ are Hamiltonian Γ-spaces, and if $J_1 : X_1 \to M$ and $J_2 : X_2 \to M$ are clean, then $\overline{X_2} \times_\Gamma X_1$ becomes a symplectic manifold. See [22] for details.

3 S^1-bundles and S^1-central extensions

In this section we recall some basic results concerning S^1-bundles and S^1-central extensions over a groupoid. For details, consult [4, 5, 18].

3.1 Integral de Rham cocycles

Let us recall some basic facts concerning singular homology. For any manifold N, we denote by $(C_\bullet(N, \mathbb{Z}), d)$ the piecewise smooth singular chain complex, and $Z_k(N, \mathbb{Z})$ the space of smooth k-cycles. For a smooth map $\phi : M \to N$, we denote by ϕ_* both the chain map from $(C_\bullet(M, \mathbb{Z}), d)$ to $(C_\bullet(N, \mathbb{Z}), d)$ and the morphism of singular homology $H_*(M, \mathbb{Z}) \to H_*(N, \mathbb{Z})$ induced by ϕ.

For any Lie groupoid $\Gamma \rightrightarrows \Gamma_0$, consider the double complex $C_\bullet(\Gamma_\bullet, \mathbb{Z})$:

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downarrow d & & \downarrow d & & \downarrow d \\
C_1(\Gamma_0, \mathbb{Z}) & \stackrel{\partial}{\leftarrow} & C_1(\Gamma_1, \mathbb{Z}) & \stackrel{\partial}{\leftarrow} & C_1(\Gamma_2, \mathbb{Z}) \\
\downarrow d & & \downarrow d & & \downarrow d \\
C_0(\Gamma_0, \mathbb{Z}) & \stackrel{\partial}{\leftarrow} & C_0(\Gamma_1, \mathbb{Z}) & \stackrel{\partial}{\leftarrow} & C_0(\Gamma_2, \mathbb{Z}),
\end{array}$$

where $\Gamma_0 = M$, and $\partial : C_k(\Gamma_p, \mathbb{Z}) \to C_k(\Gamma_{p-1}, \mathbb{Z})$ is the alternating sum of the chain maps induced by the face maps. We denote the total differential by $\delta = (-1)^p d + \partial$. Its homology will be denoted by $H_k(\Gamma_\bullet, \mathbb{Z})$. By $Z_k(\Gamma_\bullet, \mathbb{Z})$ we denote the space of k-cycles and by $[C] \in H_*(\Gamma_\bullet, \mathbb{Z})$ the class of a given cycle C. Note that $C_k(\Gamma_p, \mathbb{Z})$ is the free Abelian group generated by the piecewise smooth maps $\Delta_k \to \Gamma_p$.

The construction above can be carried out in exactly the same way replacing $\mathbb{Z}$ by $\mathbb{R}$. The corresponding homology groups are denoted by $H_k(\Gamma_\bullet, \mathbb{R})$. According to the universal-coefficient formula (see, for example, [17]), there is a canonical isomorphism

$$H_k(\Gamma_\bullet, \mathbb{R}) \simeq H_k(\Gamma_\bullet, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

There is a natural pairing between $C_\bullet(\Gamma_\bullet, \mathbb{R})$ and $C^\bullet_{dR}(\Gamma_\bullet)$ given as follows. For any generator $C : \Delta_k \to \Gamma_p$ in $C_\bullet(\Gamma_\bullet, \mathbb{Z})$,

$$\langle C, \omega \rangle = \begin{cases} \int_{\Delta_k} C^*\omega & \text{if } \omega \in \Omega^k(\Gamma_p) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

For simplicity, we will denote this pairing by $\int_C \omega$. With this notation, the pairing satisfies the following identities:

$$\int_{dC} \omega = \int_C d\omega$$

$$\int_{\partial C} \omega = \int_C \partial\omega$$

$$\int_{\delta C} \omega = \int_C \delta\omega$$

Moreover, if $\phi : G \to H$ is a groupoid homomorphism, then for any $C \in C_\bullet(G_\bullet, \mathbb{R})$ and $\omega \in C^\bullet_{dR}(H_\bullet)$

$$\int_{\phi_*(C)} \omega = \int_C \phi^*\omega. \tag{9}$$

The following result is standard (see, for example, Proposition 6.1 in [8]).

Proposition 3.1 *The pairing $H_k(\Gamma_\bullet, \mathbb{R}) \otimes H^k_{dR}(\Gamma_\bullet) \to \mathbb{R}$, $([C], [\omega]) \to \int_C \omega$ is non-degenerate.*

Let

$$Z^k_{dR}(\Gamma_\bullet,\mathbb{Z}) = \{\omega \in Z^k_{dR}(\Gamma_\bullet) | \int_C \omega \in \mathbb{Z} \text{ for any cycle } C \in Z_k(\Gamma_\bullet,\mathbb{Z})\ \}. \tag{10}$$

Elements in $Z^k_{dR}(\Gamma_\bullet,\mathbb{Z})$ are called *integral de Rham cocycles*, or simply integral cocycles.

3.2 S^1-bundles and S^1-central extensions

In this subsection, we recall some basic notations and results concerning S^1-bundles and S^1-central extensions over a Lie groupoid. For details, see [4, 5].

Definition 3.2 Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A (right) S^1-*bundle* over $\Gamma \rightrightarrows M$ is a (right) S^1-bundle P over M, together with a (left) action of Γ on P which respects the S^1-action (*i.e.*, we have $(\gamma \cdot x) \cdot t = \gamma \cdot (x \cdot t)$, for all $t \in S^1$ and all compatible pairs $(\gamma, x) \in \Gamma \times_M P$).

Let $Q \rightrightarrows P$ denote the corresponding transformation groupoid $\Gamma \times_M P \rightrightarrows P$. There is a natural groupoid homomorphism π from $Q \rightrightarrows P$ to $\Gamma \rightrightarrows M$. Of course, Q is an S^1-bundle over Γ.

A *pseudo-connection* is a 1-cochain $\theta \in C^1_{dR}(Q_\bullet)$, where $\theta \in \Omega^1(P)$ is a connection 1-form for the S^1-bundle $P \to M$. One checks that $\delta\theta \in C^2_{dR}(Q_\bullet)$ descends to a 2-cocycle in $Z^2_{dR}(\Gamma_\bullet)$. In other words, there exist unique $\omega \in \Omega^1(\Gamma)$ and $\Omega \in \Omega^2(M)$ such that

$$\delta\theta = \pi^*(\omega + \Omega).$$

Then $\omega+\Omega$ is called the *pseudo-curvature*, which is an integral 2-cocycle. Its class $[\omega+\Omega] \in H^2(\Gamma_\bullet,\mathbb{Z})$ is called the *Chern class* of the S^1-bundle P.

Proposition 3.3 *[4, 5] Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid. Assume that $\omega + \Omega \in \Omega^1(\Gamma) \oplus \Omega^2(M) \subset C^2_{dR}(\Gamma_\bullet)$ is an integral 2-cocycle. Then there exists an S^1-bundle P over $\Gamma \rightrightarrows M$ and a pseudo-connection $\theta \in \Omega^1(P)$ for the bundle $P \to M$ whose pseudo-curvature equals $\omega + \Omega$.*

Definition 3.4 Let $\Gamma \rightrightarrows M$ be a Lie groupoid. An S^1-*central extension* of $\Gamma \rightrightarrows M$ consists of

1) a Lie groupoid $R \rightrightarrows M$, together with a morphism of Lie groupoids $(\pi, \mathrm{id}) : [R \rightrightarrows M] \to [\Gamma \rightrightarrows M]$,

2) a left S^1-action on R, making $\pi : R \to \Gamma$ a (left) principal S^1-bundle. These two structures are compatible in the sense that $(s \cdot x)(t \cdot y) = st \cdot (xy)$, for all $s, t \in S^1$ and $(x, y) \in R \times_M R$.

Given a central extension R of $\Gamma \rightrightarrows M$, a *pseudo-connection* is a 2-cochain $\theta + B \in C^2_{dR}(R_\bullet)$, where $\theta \in \Omega^1(R)$ is a connection 1-form for the bundle $R \to \Gamma$ and $B \in \Omega^2(M)$. It is simple to check that $\delta(\theta + B)$ descends to a 3-cocycle in $Z^3(\Gamma_\bullet)$, *i.e.*,

$$\delta(\theta + B) = \pi^*(\eta + \omega + \Omega)$$

for some $\eta + \omega + \Omega \in Z^3(\Gamma_\bullet)$. Then $\eta + \omega + \Omega$ is an integral cocycle in $Z^3_{dR}(\Gamma_\bullet,\mathbb{Z})$, and is called the *pseudo-curvature*. Its class $[\eta + \omega + \Omega] \in H^3(\Gamma_\bullet,\mathbb{Z})$ is called the Dixmier-Douady class of R.

Proposition 3.5 *[4, 5] Assume that $\Gamma \rightrightarrows M$ is a proper Lie groupoid. Given any 3-cocycle $\eta + \omega + \Omega \in Z^3_{dR}(\Gamma_\bullet)$ such that*

1) $[\eta + \omega + \Omega]$ is integral, and

2) Ω is exact,

there exists a groupoid S^1-central extension $R \rightrightarrows M$ of the groupoid $\Gamma \rightrightarrows M$, and a pseudo-connection $\theta + B \in \Omega^1(R) \oplus \Omega^2(M)$ such that its pseudo-curvature equals $\eta + \omega + \Omega$.

3.3 Index of an S^1-bundle over a central extension

Let $R \xrightarrow{\pi} \Gamma \rightrightarrows M$ be an S^1-central extension, and $S^1 \to L \xrightarrow{p} M$ a principal S^1-bundle over the groupoid $R \rightrightarrows M$ with Chern class $[L] \in H^1(R_\bullet, S^1)$. The example below will be useful in the future.

Example 3.6 Consider, for any $k \in \mathbb{Z}$, the principal S^1-bundle $B_k : S^1 \to \cdot$ over $S^1 \rightrightarrows \cdot$, where the groupoid $S^1 \rightrightarrows \cdot$ acts on B_k by

$$\lambda \cdot z = \lambda^k z \qquad \forall \lambda \in S^1 \rightrightarrows \cdot \quad \text{and} \quad \forall z \in S^1 \to \cdot.$$

It is well-known that $H^1(S^1_\bullet, S^1) \simeq \mathbb{Z}$. Under this isomorphism, the class $[B_k]$ is simply equal to k.

It is also simple to see that the Chern class of B_k can be represented by

$$k\frac{dt}{2\pi} \in Z^1(S^1) \subset Z^2((S^1)_\bullet), \tag{11}$$

where $\frac{dt}{2\pi}$ is the normalized Haar measure on S^1.

For any $m \in M$, there exists a groupoid homomorphism f_m from $S^1 \rightrightarrows \cdot$ to $R \rightrightarrows M$ defined by

$$f_m(\lambda) = \lambda \cdot 1_m \qquad \forall \lambda \in S^1, \tag{12}$$

where $1_m \in R$ is the unit element over $m \in M$.

This homomorphism induces a map

$$f_m^* : H^1(R_\bullet, S^1) \to H^1(S^1_\bullet, S^1) \simeq \mathbb{Z}. \tag{13}$$

For a principal S^1-bundle L over $R \rightrightarrows M$, we define its *index* by

$$\mathrm{Ind}_m(L) = f_m^*([L]) \ \in H^1(S^1_\bullet, S^1) \simeq \mathbb{Z}.$$

We list some of its important properties below.

Proposition 3.7 *Let $R \xrightarrow{\pi} \Gamma \rightrightarrows M$ be an S^1-central extension, and $S^1 \to L \xrightarrow{p} M$ an (right) principal S^1-bundle over the groupoid $R \rightrightarrows M$. Then*

1. *the index is characterized by the relation*

$$f_m(\lambda) \cdot l = l \cdot \lambda^{Ind_m(L)}, \ \ \forall \lambda \in S^1, \ l \in p^{-1}(m),$$

where the dot on the left hand side denotes the R-action on L, while the dot on the right hand side refers to the S^1-action on L;

2. for any $m \in M$, the pull-back f_m^*L is isomorphic to $B_{\mathrm{Ind}_m(L)}$;

3. $\mathrm{Ind}_m(L)$ is constant on the groupoid orbits;

4. $\mathrm{Ind}_m(L)$ is constant on any connected component of M; and

5. if $\Gamma \backslash M$ is path connected, then the index $\mathrm{Ind}_m(L)$ is independent of $m \in M$.

Proof. 1) and 2) Let l be any point in the fiber $L_m = p^{-1}(m)$. For any $\lambda \in S^1$, there exists a unique $\phi(\lambda) \in S^1$ such that

$$f_m(\lambda) \cdot l = l \cdot \phi(\lambda). \tag{14}$$

The map $\lambda \to \phi(\lambda)$ does not depend on the choice of l in the fiber $p^{-1}(m)$ and is a group homomorphism from S^1 to S^1. Therefore, it is of the form $\phi(\lambda) = \lambda^k$ for some $k \in \mathbb{Z}$.

Now $f_m^*([L]) \in H^1(S^1_\bullet, S^1)$ is the Chern class associated to the pull-back of L by f_m. On the other hand, according to Eq. (14), f_m^*L is isomorphic (as a principal S^1-bundle over $S^1 \rightrightarrows \cdot$) to B_k. Therefore $k = \mathrm{Ind}_m(L)$. and Eq. (14) implies

$$f_m(\lambda) \cdot l = l \cdot \lambda^{\mathrm{Ind}_m(L)}, \quad \forall l \in p^{-1}(m). \tag{15}$$

This proves 1) and 2).

3) For any $\gamma \in R$ with $s(\gamma) = n$ and $t(\gamma) = m$, we have $\gamma 1_m = 1_n \gamma$. It follows from Eq. (12) that $\gamma f_m(\lambda) = f_n(\lambda)\gamma$. Now for any $l \in p^{-1}(m)$, we have $(\gamma f_m(\lambda)) \cdot l = (f_n(\lambda)\gamma) \cdot l$. On the one hand, we have

$$(\gamma f_m(\lambda)) \cdot l = (\gamma \cdot l) \cdot \lambda^{\mathrm{Ind}_m(L)}, \tag{16}$$

and

$$(f_n(\lambda)\gamma) \cdot l = f_n(\lambda)(\gamma \cdot l) = (\gamma \cdot l) \cdot \lambda^{\mathrm{Ind}_n(L)}. \tag{17}$$

From Eqs. (16) and (17), it follows that $\mathrm{Ind}_m(L) = \mathrm{Ind}_n(L)$.

4) It is clear from Eqs. (15) that $\mathrm{Ind}_m(L)$ depends continuously on $m \in M$. Since it is a $\mathbb{Z}$-valued function, we have $\mathrm{Ind}_m(L) = \mathrm{Ind}_n(L)$ for any pair of points $(m, n) \in M \times M$ that are in the same connected component of M.

5) follows from 3) and 4) immediately. $\Box$

3.4 Index and Chern class

From now on, we will assume that the space of orbits M/Γ is path-connected, and denote the index of L simply by $\mathrm{Ind}(L)$. Therefore we have a group homomorphism:

$$\mathrm{Ind}(L) : H^1(R_\bullet, S^1) \to \mathbb{Z}.$$

From the commutativity of the diagram

$$\begin{array}{ccc} H^1(R_\bullet, S^1) & \xrightarrow{i} & H^2(R_\bullet, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(S^1_\bullet, S^1) & \xrightarrow{i} & H^2(S^1_\bullet, \mathbb{Z}) \end{array},$$

we see that $\mathrm{Ind}(L)$ factors through $H^2(R_\bullet, \mathbb{Z}) \to \mathbb{Z}$. In the following proposition, we give an explicit formula for $\mathrm{Ind}(L)$ in terms of the Chern class.

Proposition 3.8 *Assume that $L \to M$ is a principal S^1-bundle over $R \rightrightarrows M$ with the Chern class $[\theta+\omega] \in H^2_{dR}(R_\bullet, \mathbb{Z})$, where $R \xrightarrow{\pi} \Gamma \rightrightarrows M$ is an S^1-central extension, and $\theta + \omega \in \Omega^1(R) \oplus \Omega^2(M)$. Then the index of L is given by*

$$Ind(L) = \int_{\pi^{-1}(\epsilon(m))} \theta,$$

where $\epsilon : M \to \Gamma$ is the unit map.

PROOF. Let L' be the pull-back of the principal S^1-bundle L via the homomorphism $f_m : S^1 \to R$. The Chern class of L' is the pull-back of the Chern class of L, *i.e.*, the class defined by $f_m^*\theta + f_m^*\omega \in C^2_{dR}(S^1_\bullet)$. Since $f_m^*\omega$ is a 2-form over a point, it vanishes and therefore the Chern class of L' is represented by $f_m^*\theta \in \Omega^1(S^1)$.

By Proposition 3.7, L' is isomorphic to $B_{\mathrm{Ind}(L)}$. According to Eq. (11), the identity $f_m^*\,\theta = \mathrm{Ind}(L)\,\frac{dt}{2\pi} + \delta g = \mathrm{Ind}(L)\,\frac{dt}{2\pi} + dg$ holds for some function $g \in C^\infty(S^1, \mathbb{R})$.

Now since f_m is a bijection from S^1 to $\pi^{-1}(\epsilon(m))$, we have

$$\int_{\pi^{-1}(\epsilon(m))} \theta = \int_{S^1} f_m^*\theta.$$

Therefore

$$\int_{\pi^{-1}(\epsilon(m))} \theta = \int_{S^1} f_m^*\theta = \mathrm{Ind}(L) \int_{S^1} \frac{dt}{2\pi} + \int_{S^1} dg = \mathrm{Ind}(L).$$

□

Recall that a line bundle $L \to M$ over $R \rightrightarrows M$ is called a (Γ, R)-twisted line bundle if $\ker \pi \cong M \times S^1$ acts on L by scalar multiplication, where S^1 is identified with the unit circle in $\mathbb{C}$ [18]. The following corollary is an immediate consequence of Proposition 3.8 and Proposition 3.7.

Corollary 3.9 *Under the hypotheses of Proposition 3.8, $L \to M$ defines a (Γ, R)-twisted line bundle if and only if*

$$\int_{\pi^{-1}(\epsilon(m))} \theta = 1.$$

4 Prequantization of classical intertwiner spaces

4.1 Compatible prequantizations

Definition 4.1 A prequantization of a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$ consists of an S^1-central extension $R \xrightarrow{\pi} \Gamma \rightrightarrows M$ together with a pseudo-connection $\theta + B \in \Omega^1(R) \oplus \Omega^2(M)$ such that

$$\delta(\theta + B) = \pi^*(\omega + \Omega). \tag{18}$$

According to Proposition 3.3, if $\Gamma \rightrightarrows M$ is a proper Lie groupoid, a prequantization exists if and only if $(\Gamma \rightrightarrows M, \omega + \Omega)$ is exact and $\omega + \Omega$ is an integral 3-cocycle. A pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$ is said to be *integral* if $\omega + \Omega$ is an integral cocycle.

Definition 4.2 Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an exact pre-quasi-symplectic groupoid, and $(R \to \Gamma \rightrightarrows M, \theta + B)$ a prequantization. Assume that $(X \xrightarrow{J} M, \omega_X)$ is a pre-Hamiltonian Γ-space. A compatible prequantization of X consists of an S^1-bundle $\phi : L \to X$ with a connection 1-form $\theta_L \in \Omega^1(L)$ such that

1. $\widetilde{J} = J \circ \phi : L \longrightarrow M$ is a left R-space and the action satisfies:
$$(s \cdot \kappa)(t \cdot x) = st \cdot (\kappa x),$$
for all $s, t \in S^1$ and $(\kappa, x) \in R \times_M X$ a compatible pair;

2. the 1-form $(\theta, \theta_L, -\theta_L) \in \Omega^1(R \times L \times \overline{L})$ vanishes on the graph of the action
$$\Xi = \{(\kappa, l, \kappa l) | \kappa \in R, l \in L \text{ compatible pairs}\};$$
and

3. $d\theta_L = \phi^*(J^*B - \omega_X)$.

Note that the second condition above is equivalent to saying that $(R \times L \times \overline{L})/T^2 \xrightarrow{p} \Gamma \times X \times X$ with $p([\kappa, l, m]) = (\pi(\kappa), \phi(l), \phi(m))$ is a flat S^1-bundle with the connection $\bar{\Theta}$, which is the 1-form on $(R \times L \times \overline{L})/T^2$ naturally induced from $\Theta = (\theta, \theta_L, -\theta_L) \in \Omega^1(R \times L \times L)$ (see [21]).

Example 4.3 If Γ is the symplectic groupoid $(T^*G \rightrightarrows \mathfrak{g}^*, \omega)$, where $\omega \in \Omega^2(T^*G)$ is the canonical cotangent symplectic 2-form, a prequantization of Γ can be taken to be $R \cong T^*G \times S^1 \to T^*G$, the trivial S^1-bundle and $\theta = \theta_{T^*G} + dt$, where $\theta_{T^*G} \in \Omega^1(T^*G)$ is the Liouville 1-form and t is the natural coordinate on S^1. A Hamiltonian Γ-space is a Hamiltonian G-space $J : X \to \mathfrak{g}^*$ in the usual sense. It is simple to see that a compatible pre-quantization is a G-equivariant prequantization of X, which always exists when G is connected and simply connected [10].

More generally, the following result was proved in [21] (the theorem was stated for the symplectic case, but it is valid for the presymplectic case as well).

Proposition 4.4 *Let $(\Gamma \rightrightarrows M, \omega)$ be an s-connected and s-simply connected pre-symplectic groupoid, and $(X \xrightarrow{J} M, \omega_X)$ a pre-Hamiltonian space. If both ω and ω_X represent integral cohomology classes in $H^2_{dR}(\Gamma)$ and $H^2_{dR}(X)$ respectively, then there exists a compatible prequantization.*

For a given pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$ and a prequantization $(R \to \Gamma \rightrightarrows M, \theta + B)$, let $\Gamma \times_M X \rightrightarrows X$ be the transformation groupoid as in Eq. (4). By pulling back the central extension $R \to \Gamma \rightrightarrows M$ via J, one obtains a central extension of groupoids $R \times_M X \to \Gamma \times_M X \rightrightarrows X$. Here $R \times_M X$ is again a transformation groupoid, where R acts on X by projecting R to Γ and using the given Γ-action on X.

By abuse of notation, we still use J to denote the projection $R \times_M X \to R$. Therefore we have the following homomorphism of S^1-central extensions of groupoids:

$$\begin{tikzcd} R \times_M X \arrow[r, "J"] \arrow[d] & R \arrow[d] \\ \Gamma \times_M X \arrow[r, "J"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[r, "J"] & M \end{tikzcd} \tag{19}$$

Remark 4.5 Note that Proposition 2.3 implies that the Dixmier-Douady class of $R \times_M X \to \Gamma \times_M X \rightrightarrows X$ vanishes. If $\Gamma \rightrightarrows M$ is a proper groupoid, so is $\Gamma \times_M X \rightrightarrows X$. Therefore $R \times_M X \to \Gamma \times_M X \rightrightarrows X$ defines a trivial gerbe. According to Proposition 4.2 of [4], there exists an S^1-bundle $E \to X$ such that $R \times_M X \cong s^*E \otimes t^*\overline{E}$ as a central extension.

Proposition 4.6 *Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an exact pre-quasi-symplectic groupoid, and $(R \to \Gamma \rightrightarrows M, \theta + B)$ its prequantization. Assume that $(X \xrightarrow{J} M, \omega_X)$ is a pre-Hamiltonian Γ-space. Then $(L \xrightarrow{\phi} X, \theta_L)$ is a compatible prequantization of X if and only if (the associated line bundle of) $\phi : L \to X$ is a twisted line bundle over $R \times_M X \to \Gamma \times_M X \rightrightarrows X$ with the pseudo-connection and the pseudo-curvature being given by θ_L and $J^*\theta + (J^*B - \omega_X) \in \Omega^1(R \times_M X) \oplus \Omega^2(X)$, respectively.*

Proof. Given a compatible prequantization $L \xrightarrow{\phi} X$, define an action of $R \times_M X \rightrightarrows X$ on L by $(\kappa, \phi(l)) \cdot l = \kappa l$, where $\kappa \in R$ and $l \in L$ are compatible pairs. It is simple to check that all the compatibility conditions are satisfied so that $L \xrightarrow{\phi} X$ is a twisted line bundle over the central extension $R \times_M X \to \Gamma \times_M X \rightrightarrows X$. It is simple to see that the corresponding transformation groupoid $(R \times_M X) \times_X L \rightrightarrows L$ is isomorphic to the transformation groupoid $R \times_M L \rightrightarrows L$. Moreover, it is simple to see that Condition (2) of Definition 4.2 implies that

$$\partial \theta_L = \phi^* J^* \theta, \tag{20}$$

where, by abuse of notation, we use ϕ to denote the Lie groupoid homomorphism:

$$\begin{tikzcd} R \times_M L \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & R \times_M X \arrow[d, shift left] \arrow[d, shift right] \\ L \arrow[r, "\phi"] & X \end{tikzcd} \tag{21}$$

and $\partial \theta_L$ is with respect to the groupoid $R \times_M L \rightrightarrows L$. Therefore we have

$$\delta \theta_L = \partial \theta_L + d\theta_L = \phi^*(J^*\theta + J^*B - \omega_X).$$

The converse can be proved by working backwards. $\square$

As an immediate consequence, we have

Corollary 4.7 *Under the hypotheses of Proposition 4.6 and assuming that $\Gamma \rightrightarrows M$ is proper, for a pre-Hamiltonian Γ-space $(X \xrightarrow{J} M, \omega_X)$, a compatible prequantization exists if and only if $J^*(\theta + B) - \omega_X$ is an integral 2-cocycle in $Z^2_{dR}((R \times_M X)_\bullet, \mathbb{Z})$.*

Proof. One direction is obvious by Proposition 4.6.

For the other direction, note that Proposition 2.3 implies that $J^*(\theta + B) - \omega_X$ is always a 2-cocycle since

$$\delta(J^*(\theta + B) - \omega_X) = J^*\delta(\theta + B) - \pi^*\delta\omega_X = J^*\pi^*(\omega + \Omega) - \pi^*J^*(\omega + \Omega) = 0.$$

Here we have used Eqs. (18) and (6). If $J^*(\theta+B)-\omega_X$ is an integral cocycle in $Z^2_{dR}((R \times_M X)_\bullet, \mathbb{Z})$, according to Proposition 3.3, there exists an S^1-bundle $L \to X$ over $R \rightrightarrows X$ and a pseudo-connection $\theta_L \in \Omega^1(L)$ whose pseudo-curvature equals to $J^*\theta + (J^*B - \omega_X)$. According to Corollary 3.9, one sees that (the associated line bundle of) L is indeed a twisted line bundle over $R \times_M X \to \Gamma \times_M X \rightrightarrows X$. Then $L \to X$ is a compatible prequantization by Proposition 4.6. $\square$

4.2 Prequantization of classical intertwiner spaces

We are now ready to state the main theorem of this section.

Theorem 4.8 *Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an exact pre-quasi-symplectic groupoid, and $(R \to \Gamma \rightrightarrows M, \theta + B)$ a prequantization. Assume that $(X_k \stackrel{J_k}{\to} M, \omega_k)$, $k = 1, 2$, are pre-Hamiltonian Γ-spaces, $\Gamma \rightrightarrows M$ acts freely on $\overline{X_2} \times_M X_1$, and $\overline{X_2} \times_\Gamma X_1 = \Gamma\backslash(\overline{X_2} \times_M X_1)$ is a smooth manifold. Let $(L_k \stackrel{\phi_k}{\to} X_k, \theta_k)$, be a compatible prequantization of X_k for $k = 1, 2$. Then*

$$\phi : R\backslash(L_2 \times_M \overline{L_1}) \to \overline{X_2} \times_\Gamma X_1,\ \phi[l_2, l_1] = [\phi_2(l_2), \phi_1(l_1)],$$

with the S^1-action $\lambda \cdot [l_2, l_1] = [\lambda \cdot l_2, l_1]$, $\lambda \in S^1$, is an S^1-principal bundle. Moreover, $i^*(\theta_2, -\theta_1)$ descends to a connection 1-form on $R\backslash(L_2 \times_M \overline{L_1})$, which defines a prequantization of the classical intertwiner space $\overline{X_2} \times_\Gamma X_1$. Here $i : L_2 \times_M \overline{L_1} \to L_2 \times \overline{L_1}$ is the natural embedding.

Proof. One checks directly that $\phi : R\backslash(L_2 \times_M L_1) \to \overline{X_2} \times_\Gamma X_1$ is an S^1-bundle. Now let $R \rightrightarrows M$ act on $L_2 \times_M L_1$ diagonally. We have

$$\partial i^*(\theta_2, -\theta_1) = i^*(\partial\theta_2, -\partial\theta_1) = i^*(\phi_2^* J_2^* \theta, -\phi_1^* J_1^* \theta) = 0.$$

Hence $i^*(\theta_2, -\theta_1)$ descends to a 1-form on the quotient space $R\backslash(\overline{L_2} \times_M L_1)$, which can be easily seen to be a connection 1-form. Now

$$d(i^*(\theta_2, -\theta_1)) = i^*(d\theta_2, -d\theta_1) = i^*(\phi_2^*(J_2^* B - \omega_2), \phi_1^*(J_1^* B - \omega_1)) = i^*(\phi_2 \times \phi_1)^*(-\omega_2, \omega_1),$$

where in the last equality we used the relation $J_1 \circ \phi_1 = J_2 \circ \phi_2$ on $L_2 \times_M L_1$. Here ϕ_i and J_k, $k = 1, 2$ are groupoid homomorphisms:

$$\begin{array}{ccccc} R \times_M L_k & \xrightarrow{\phi_k} & R \times_M X_k & \xrightarrow{J_k} & R \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ L_k & \xrightarrow{\phi_k} & X_k & \xrightarrow{J_k} & M \end{array} \tag{22}$$

and i is the groupoid homomorphism:

$$\begin{array}{ccc} R \times_M (L_2 \times_M L_1) & \xrightarrow{i} & (R \times_M L_2) \times (R \times_M L_1)\ . \\ \downdownarrows & & \downdownarrows \\ L_2 \times_M L_1 & \xrightarrow{i} & L_2 \times L_1 \end{array} \tag{23}$$

This completes the proof. $\square$

4.3 Morita equivalence

Definition 4.9 Pre-quasi-symplectic groupoids $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are said to be *Morita equivalent* if there exists a Morita equivalence bimodule $G_0 \stackrel{\rho}{\leftarrow} X \stackrel{\sigma}{\to} H_0$ between the Lie groupoids G and H, together with a 2-form $\omega_X \in \Omega^2(X)$ such that $(X \stackrel{\rho\times\sigma}{\to} G_0 \times \overline{H_0}, \omega_X)$ is a pre-Hamiltonian $G \times \overline{H}$-space, where the $G \times \overline{H}$-action on X is given by $(g, h) \cdot x = gxh^{-1}$ for all compatible triples $g \in G$, $h \in H$ and $x \in X$.

One easily checks that this is indeed an equivalence relation among pre-quasi-symplectic groupoids.

Let $Q \rightrightarrows X$ be the transformation groupoid

$$Q : (G \times \overline{H}) \times_{(G_0 \times \overline{H_0})} X \rightrightarrows X.$$

Then the natural projections $\mathrm{pr}_1 : Q \to G$ and $\mathrm{pr}_2 : Q \to \overline{H}$ are groupoid homomorphisms. As an immediate consequence of Proposition 2.3, we have the following identity

$$\mathrm{pr}_1^*(\omega_G + \Omega_G) - \mathrm{pr}_2^*(\omega_H + \Omega_H) = \delta\omega_X.$$

Note that the axioms of Morita equivalence of Lie groupoids assure that, as groupoids, $Q \cong G[X]$ and $Q \cong H[X]$ (see the proof of Proposition 4.5 of [22]), where $G[X] \rightrightarrows X$ and $H[X] \rightrightarrows X$ are the pull-back groupoids of G and H using ρ and σ, respectively.

Recall that for a given Lie groupoid $\Gamma \rightrightarrows M$, two cohomologous 3-cocycles $\omega_i + \Omega_i \in \Omega^2(\Gamma) \oplus \Omega^3(M)$, $i = 1, 2$, are said to differ by a *gauge transformation of the first type* if

$$(\omega_1 + \Omega_1) - (\omega_2 + \Omega_2) = \delta B$$

for some $B \in \Omega^2(M)$.

By a *Morita morphism* from the pre-quasi-symplectic groupoid $(\Gamma' \rightrightarrows M', \omega' + \Omega')$ to $(\Gamma \rightrightarrows M, \omega + \Omega)$, we mean a Morita morphism of the Lie groupoid $p : \Gamma' \to \Gamma$ (*i.e.* Γ' is isomorphic to the pullback groupoid $\Gamma[M'] \rightrightarrows M'$) such that $\omega' + \Omega'$ and $p^*\omega + p^*\Omega$ differ by a gauge transformation of the first type.

The following result gives a more intuitive explanation of Morita equivalence.

Proposition 4.10 *Two pre-quasi-symplectic groupoids are Morita equivalent if and only if there exists a third pre-quasi-symplectic groupoid together with a Morita morphism to each of them.*

Corollary 4.11 *For two Morita equivalent pre-quasi-symplectic groupoids, if one is integral, so is the other.*

Therefore Morita equivalence induces an equivalence relation among integral pre-quasi-symplectic groupoids.

One of the most important features of Morita equivalent pre-quasi-symplectic groupoids is the following

Theorem 4.12 *Suppose that $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are Morita equivalent pre-quasi-symplectic groupoids with an equivalence bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$. Then:*

1. *Corresponding to any pre-Hamiltonian G-space $J_F : F \to G_0$, there is a unique (up to isomorphism) pre-Hamiltonian H-space $J_E : E \to H_0$ such that F and E are a pair of related pre-Hamiltonian spaces and vice versa.*

2. *Let $J_{F_i} : F_i \to G_0$, $i = 1, 2$, be pre-Hamiltonian G-spaces and $J_{E_i} : E_i \to H_0$, $i = 1, 2$, their related pre-Hamiltonian H-spaces. Then $\overline{F_2} \times_G F_1$ and $\overline{E_2} \times_H E_1$ are diffeomorphic as presymplectic manifolds (in the sense that if one is smooth so is the other).*

Proof. This was proved in [22] for quasi-symplectic groupoids and their Hamiltonian spaces. One can prove this theorem in a similar fashion (in fact in a simpler way by using Proposition 2.4). We will leave the details to the reader. $\Box$

We now can introduce Morita equivalence for the prequantization of pre-quasi-symplectic groupoids.

Definition 4.13 Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ be Morita equivalent integral exact pre-quasi-symplectic groupoids with an equivalence bimodule $(G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0,\ \omega_X)$. We say their prequantizations $(R_G \to G \rightrightarrows G_0, \theta_G + B_G)$ and $(R_H \to H \rightrightarrows H_0, \theta_H + B_H)$ are Morita equivalent if X admits a compatible prequantization $(Z \to X, \theta_Z)$ with respect to the prequantization of the pre-quasi-symplectic groupoid $(R_G \times \overline{R_H})/S^1 \to G \times \overline{H} \rightrightarrows G_0 \times \overline{H_0}, (\theta_G + \overline{\theta_H}) + (B_G + \overline{B_H}))$.

It is simple to see that $G_0 \leftarrow Z \rightarrow H_0$ is an equivalence bimodule of central extensions in the sense of Definition 2.11 [18].

Remark 4.14

1. Note that prequantizations can be Morita equivalent as central extensions, but not Morita equivalent as prequantizations. The former one simply means that they correspond to isomorphic S^1-gerbes, and up to a torsion, are determined by their Dixmier-Douady classes.
2. It would be interesting to investigate the following question: given two Morita equivalent pre-quasi-symplectic groupoids and a prequantization of one of them, is it possible to construct a Morita equivalent prequantization for the other pre-quasi-symplectic groupoid?

A useful feature of Morita equivalence is that it gives a recipe which allows us to construct compatible prequantizations.

Theorem 4.15 *For Morita equivalent prequantizations of pre-quasi-symplectic groupoids, there is an equivalence of categories of compatible prequantizations of pre-Hamiltonian spaces.*

Proof. Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ be Morita equivalent integral exact pre-quasi-symplectic groupoids with an equivalence bimodule $(G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0,\ \omega_X)$, and $(R_G \to G \rightrightarrows G_0, \theta_G + B_G)$ and $(R_H \to H \rightrightarrows H_0, \theta_H + B_H)$ be Morita equivalent prequantizations given by $(Z \to X, \theta_Z)$. Assume that $J : F \to G_0$ is a pre-Hamiltonian G-space and $(L \to F, \theta_L)$ a compatible prequantization. It is known that the corresponding pre-Hamiltonian H-space is $E := \overline{X} \times_G F \overset{J'}{\rightarrow} H_0$, where $J' : E \to H_0$ and the H-action on E are defined by $J'([x, f]) = \sigma(x)$ and $h \cdot [x, f] = [x \cdot h^{-1}, f]$, respectively.

Let $L' = Z \times_{R_G} \overline{L}$. Then it is clear that L' is an S^1-bundle over E, and R_H acts on L' equivariantly. It is simple to check that $i^*(\theta_Z, -\theta_L)$, where $i : Z \times_{G_0} L \to Z \times L$, descends to a 1-form on the quotient space $Z \times_{R_G} \overline{L}$ which is indeed a connection 1-form $\theta_{L'}$ on L'. It is routine to check that $(L' \to E, \theta_{L'})$ is a compatible prequantization of the pre-Hamiltonian H-space $J' : E \to H_0$.

The inverse functor can be constructed in a similar fashion. $\Box$

Remark 4.16 The above theorem indicates a useful method which enables one to transform prequantizations of Hamiltonian LG-spaces to prequantizations of quasi-Hamiltonian G-spaces of

AMM and vice-versa. The latter is understood as a compatible prequantization corresponding to the quasi-symplectic groupoid $(G\times G)[\mathcal{U}] \rightrightarrows \coprod U_i$, which is the pull-back quasi-symplectic groupoid of the AMM quasi-symplectic groupoid using an open covering $\mathcal{U} = (U_i)_{i\in I}$ of G (see [14], for instance, for an explicit construction). It is known that $(G \times G)[\mathcal{U}] \rightrightarrows \coprod U_i$ is Morita equivalent to the symplectic groupoid $(LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}, \omega_{LG\times L\mathfrak{g}})$ according to Proposition 4.26 [22]. The question is therefore boiled down to the construction of a compatible prequantization of the Morita equivalence Hamiltonian bimodule.

5 Integral pre-Hamiltonian Γ-spaces

The main purpose of this section is to give a geometric integrality condition which guarantees the existence of a prequantization of a pre-Hamiltonian Γ-space.

5.1 Integrality condition

Lemma 5.1 *Let $J : G \to H$ be a groupoid homomorphism. By $Ker(J_*)$ we denote the kernel of $J_* : H_2(G_\bullet, \mathbb{Z}) \to H_2(H_\bullet, \mathbb{Z})$. Let $\omega \in Z^2_{dR}(G_\bullet)$. The following conditions are equivalent:*

1. *there exists $\Xi \in Z^2_{dR}(H_\bullet)$ such that*
$$\omega + J^*\Xi \in Z^2_{dR}(G_\bullet, \mathbb{Z});$$
2. *for any $C \in Z_2(G_\bullet, \mathbb{Z})$ with $[C] \in Ker(J_*)$, we have*
$$\int_C \omega \in \mathbb{Z}.$$

Proof. 1) ⇒ 2). By definition, we have for any $C \in Z_2(G_\bullet, \mathbb{Z})$,
$$\int_C (\omega + J^*\Xi) \in \mathbb{Z}. \tag{24}$$
From Eq. (9), we also have
$$\int_C (\omega + J^*\Xi) = \int_C \omega + \int_{J_*(C)} \Xi.$$
If $[J_*(C)] = 0$, *i.e.*, $J_*(C) = \delta D$ for some $D \in C_3(H_\bullet, \mathbb{Z})$, then
$$\int_C (\omega + J^*\Xi) = \int_C \omega + \int_{\delta D} \Xi = \int_C \omega + \int_D \delta\Xi = \int_C \omega$$
since $\delta\Xi = 0$. Therefore $\int_C \omega \in \mathbb{Z}$.

2) ⇒ 1). Since there exists a $\mathbb{Z}$-submodule $\mathcal{H}$ in $H_2(G_\bullet, \mathbb{Z})$ such that $H_2(G_\bullet, \mathbb{Z}) = \mathcal{H} \oplus \mathrm{Ker}(J_*)$, the $\mathbb{Z}$-map
$$f : \mathrm{Ker}(J_*) \to \mathbb{Z},\ f([C]) = \int_C \omega,\ \ \forall [C] \in \mathrm{Ker}(J_*),$$

can be extended to a $\mathbb{Z}$-map $\widetilde{f} : H_2(G_\bullet, \mathbb{Z}) \to \mathbb{Z}$. According to Proposition 3.1, there exists $\omega' \in Z^2_{dR}(G_\bullet)$ such that

$$\widetilde{f}([C]) = \int_C \omega', \quad \forall [C] \in H_2(G_\bullet, \mathbb{Z}).$$

By Eq. (10), ω' is an integral cocycle in $Z^2_{dR}(G_\bullet, \mathbb{Z})$. Moreover, we have

$$\int_C (\omega' - \omega) = 0, \quad \forall C \in Z_2(G_\bullet, \mathbb{Z}) \text{ such that } [C] \in \mathrm{Ker}(J_*). \tag{25}$$

Since $J_* : H_2(G_\bullet, \mathbb{R}) \to H_2(H_\bullet, \mathbb{R})$ is dual to $J^* : H^2_{dR}(H_\bullet) \to H^2_{dR}(G_\bullet)$, we have $\mathrm{Ker}(J_*)^\perp = \mathrm{Im}(J^*)$. Therefore $[\omega' - \omega] = J^*[\Xi]$ for some $\Xi \in Z^2_{dR}(\Gamma_\bullet)$. This proves 1). □

Definition 5.2 Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be a pre-quasi-symplectic groupoid. A pre-Hamiltonian Γ-space $(X \to M, \omega_X)$ is said to satisfy the *integrality condition* if for any $C \in Z_2((\Gamma \times_M X)_\bullet, \mathbb{Z})$ and any $D \in C_3(\Gamma_\bullet, \mathbb{Z})$

$$\delta D = J_*(C) \qquad \Rightarrow \qquad \int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}. \tag{26}$$

In this case, we also say that the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition.

Remarks 5.3
1. By taking $C = 0$, Eq. (26) implies that $\int_D (\omega + \Omega) \in \mathbb{Z}, \quad \forall D \in Z_3(\Gamma_\bullet, \mathbb{Z})$. That is, $\omega + \Omega$ must be an integral 3-cocycle and therefore $(\Gamma \rightrightarrows M, \omega + \Omega)$ must be an integral pre-quasi-symplectic groupoid.
2. If $\omega + \Omega$ is a 3-coboundary δK, then the integrality condition is equivalent to

$$\int_C (\omega_X - J^* K) \in \mathbb{Z}, \quad \forall C \in Z_2((\Gamma \times_M X)_\bullet, \mathbb{Z}) \text{ such that } J_*[C] = 0. \tag{27}$$

From now on, we shall always assume that $(\Gamma \rightrightarrows M, \omega + \Omega)$ is an integral pre-quasi-symplectic groupoid. The following lemma indicates that it is sufficient to require that both sides of Eq. (26) hold for a single representative (C, D) in every class of $\mathrm{Ker}(J_*)$.

Lemma 5.4 *Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an integral pre-quasi-symplectic groupoid. A pre-Hamiltonian Γ-space $(X \to M, \omega_X)$ satisfies the integrality condition if and only if for any class $\mathfrak{c} \in KerJ_*$, there exists $C \in Z_2((\Gamma \times_M X)_\bullet, \mathbb{Z})$ and $D \in C_3(\Gamma_\bullet, \mathbb{Z})$ with $\mathfrak{c} = [C]$ and $J_*(C) = \delta D$ such that*

$$\int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}.$$

Proof. Let $C' \in Z_2((\Gamma \times_M X)_\bullet, \mathbb{Z})$ and $D' \in C_3(\Gamma_\bullet, \mathbb{Z})$ be any pair satisfying $J_*(C') = \delta D'$. Then $[C'] \in \mathrm{Ker}(J_*)$. By assumption, there exists a pair (C, D) such that $[C] = [C']$, $J_*(C) = \delta D$, and

$\int_C \omega_X - \int_D(\omega+\Omega) \in \mathbb{Z}$. Assume that $C = C' + \delta E$ for some $E \in C_3((\Gamma \times_M X)_\bullet, \mathbb{Z})$. Then we have

$$\begin{aligned}
& \int_{C'} \omega_X - \int_{D'} (\omega+\Omega) - \Big(\int_C \omega_X - \int_D (\omega+\Omega)\Big) \\
= & -\int_{\delta E} \omega_X + \int_D (\omega+\Omega) - \int_{D'} (\omega+\Omega) \\
= & -\int_E \delta\omega_X + \int_D (\omega+\Omega) - \int_{D'} (\omega+\Omega) \\
= & -\int_E J^*(\omega+\Omega) + \int_D (\omega+\Omega) - \int_{D'} (\omega+\Omega) \\
= & \int_{D-J_*(E)-D'} (\omega+\Omega).
\end{aligned}$$

Since $\delta(D - J_*(E) - D') = J_*(C - C' - \delta E) = 0$ and $\omega + \Omega$ is an integral cocycle, it follows that $\int_{D-J_*(E)-D'}(\omega+\Omega) \in \mathbb{Z}$. This completes the proof. $\square$

Now assume that $(\Gamma \rightrightarrows M, \omega + \Omega)$ is an integral exact pre-quasi-symplectic groupoid, and $R \to \Gamma \rightrightarrows M$ is a prequantization. Let $\theta + B \in \Omega^1(R) \oplus \Omega^2(M)$ be a pseudo-connection satisfying Eq. (18). In order to fix the notation, recall that we have the following commutative diagram of groupoid homomorphisms

$$\begin{array}{ccc}
R \times_M X & \stackrel{J}{\to} & R \\
\downarrow \pi & & \downarrow \pi \\
\Gamma \times_M X & \stackrel{J}{\to} & \Gamma
\end{array} \tag{28}$$

where the horizontal arrows are projections.

Lemma 5.5 *Assume that $C' \in Z_2((R \times_M X)_\bullet, \mathbb{Z})$ satisfies $J_*(C') = kZ + \delta D'$ for some $D' \in C_3(R_\bullet, \mathbb{Z})$. Let $C = \pi_*(C')$ and $D = \pi_*(D')$. Then*

$$\int_C \omega_X - \int_D (\omega+\Omega) = k + \int_{C'} \big(\omega_X - J^*(\theta+B)\big), \tag{29}$$

where $Z \in Z_1(R_\bullet, \mathbb{Z})$ is the 1-cycle defined by Eq. (41).

Proof. First, since $\pi : R \times_M X \to \Gamma \times_M X$ reduces to the identity map when being restricted to the unit spaces, we have

$$\int_{C'} \omega_X = \int_{\pi_*(C')} \omega_X = \int_C \omega_X. \tag{30}$$

Now by Eq. (9), we have

$$\int_{C'} J^*(\theta+B) = \int_{J_*(C')} (\theta+B) = k\int_Z (\theta+B) + \int_{\delta D'} (\theta+B). \tag{31}$$

According to Lemma 6.1, $\int_Z(\theta+B)=\int_Z\theta=1$. Therefore

$$\begin{aligned}
\int\limits_{C'} J^*(\theta+B) &= k+\int\limits_{\delta D'}(\theta+B) \\
&= k+\int\limits_{D'}\delta(\theta+B) && \text{(by Eq. (18))} \\
&= k+\int\limits_{D'}\pi^*(\omega+\Omega) && \text{(by Eq. (9))} \\
&= k+\int\limits_{D}(\omega+\Omega).
\end{aligned}$$

Hence it follows that

$$\int\limits_{C}\omega_X-\int\limits_{D}(\omega+\Omega)=k+\int\limits_{C'}(\omega_X-J^*(\theta+\Omega)).$$

□

The following proposition gives a useful characterization of integrality condition.

Proposition 5.6 *Let $(\Gamma\rightrightarrows M,\omega+\Omega)$ be an integral pre-quasi-symplectic groupoid, and $(R\to\Gamma\rightrightarrows M,\theta+B)$ a prequantization. Assume that $(X\xrightarrow{J}M,\omega_X)$ is a pre-Hamiltonian Γ-space. Then the following conditions are equivalent.*

1. *There exists a 2-cocycle $\Xi\in Z^2_{dR}(\Gamma_\bullet)$ such that*
$$\omega_X-J^*(\theta+B)-J^*\pi^*\Xi\in Z^2_{dR}\big((R\times_M X)_\bullet,\mathbb{Z}\big).$$
2. *For any cycle $C'\in Z_2((R\times_M X)_\bullet,\mathbb{Z})$ such that $[C']\in Ker(\pi_*\circ J_*)$, we have*
$$\int\limits_{C'}\big(\omega_X-J^*(\theta+B)\big)\in\mathbb{Z}.$$
3. *The pair $(\omega_X,\omega+\Omega)$ satisfies the integrality condition.*

Proof. 1) $\iff$ 2). follows from Lemma 5.1.

2) $\Rightarrow$ 3). Any class in Ker$J_*\subset H^2((\Gamma\times_M X)_\bullet,\mathbb{Z})$ can be represented by a 2-cocycle of the form $C=\pi_*(C')$ where $C'\in Z_2((R\times_M X)_\bullet,\mathbb{Z})$. Then $[C']$ is in the kernel of $J_*\circ\pi_*=\pi_*\circ J_*$. It thus follows that $[J_*(C')]\in\mathrm{Ker}(\pi_*)$. By Lemma 6.1, $[J_*(C')]=k[Z]$ for some $k\in\mathbb{Z}$ where $Z\in C_1(R_\bullet,\mathbb{Z})$ is defined by Eq. (41). In other words, there exists $D'\in C_3(R_\bullet,\mathbb{Z})$ such that $J_*(C')=kZ+\delta D'$. Let $D=\pi_*(D')$. One can easily see that $\delta D=\pi_*(J_*(C'))=J_*(C)$. Then by Lemma 5.5, we have $\int_C\omega_X-\int_D(\omega+\Omega)\in\mathbb{Z}$. By Lemma 5.4, this implies that the pair $(\omega_X,\omega+\Omega)$ satisfies the integrality condition.

3) $\Rightarrow$ 2). Let $C'\in Z_2((R\times X)_\bullet,\mathbb{Z})$ be any cycle whose class is in the kernel of $\pi_*\circ J_*$. Since $[J_*(C')]\in\mathrm{Ker}(\pi_*)$, Lemma 6.1 implies that there exists $k\in\mathbb{Z}$ and $D'\in C_3(R_\bullet,\mathbb{Z})$ such that

$$J_*(C')=kZ+\delta D'. \tag{32}$$

Therefore, by Eq. (29), we have $\int_{C'}(\omega_X - J^*(\theta+B)) = -k + \int_C \omega_X - \int_D(\omega+\Omega)$, where $C = \pi_*(C')$ and $D = \pi_*(D')$. By applying π_* to Eq. (32), one finds that $J_*(C) = \delta D$. Since $(\omega_X, \omega+\Omega)$ satisfies the integrality condition, it thus follows that $\int_{C'}(\omega_X - J^*(\theta+B)) \in \mathbb{Z}$. $\square$

As an immediate consequence, we obtain the following main result of the section.

Theorem 5.7 *Let $(\Gamma \rightrightarrows M, \omega+\Omega)$ be an exact proper pre-quasi-symplectic groupoid, and $(X \xrightarrow{J} M, \omega_X)$ a pre-Hamiltonian Γ-space. Then there exists a compatible prequantization $R \to \Gamma \rightrightarrows M$ and $L \to X$ if and only if the pair $(\omega_X, \omega+\Omega)$ satisfies the integrality condition of Eq. (26).*

PROOF. Assume that $(R \to \Gamma \rightrightarrows M, \theta+B)$ and $(L \to X, \theta_L)$ are a pair of compatible prequantizations. By Corollary 4.7, we have $\omega_X - J^*(\theta+B) \in Z^2_{dR}((R \times_M X)_\bullet, \mathbb{Z})$. Hence $(\omega_X, \omega+\Omega)$ satisfies the integrality condition according to Proposition 5.6.

Conversely, assume that $(\omega_X, \omega+\Omega)$ satisfies the integrality condition. Then $\omega+\Omega$ must be an integral cocycle. Let $(R \to \Gamma \rightrightarrows M, \theta+B)$ be a prequantization, which always exists since Γ is proper. Again according to Proposition 5.6, there exists a 2-cocycle $\Xi \in Z^2_{dR}(\Gamma_\bullet)$ such that $\omega_X - J^*(\theta+B) - J^*\pi^*\Xi \in Z^2_{dR}((R \times_M X)_\bullet, \mathbb{Z})$. Since Γ is proper, Ξ is cohomologous to $\alpha + B_0$, where $\alpha \in \Omega^1(\Gamma)$ is a closed 1-form and $B_0 \in \Omega^2(M)$. Then $\theta' + B' := (\theta + \pi^*\alpha) + (B + B_0)$ is clearly also a pseudo-connection and $\omega_X - J^*(\theta'+B') \in Z^2_{dR}((R \times_M X)_\bullet, \mathbb{Z})$. From Corollary 4.7, it follows that $(X \xrightarrow{J} M, \omega_X)$ admits a compatible prequantization $(L \to X, \theta_L)$. $\square$

5.2 Integral quasi-Hamiltonian G-spaces

In this subsection, G is a connected and simply-connected compact Lie group and 1 denotes the unit of G. We intend to study the case where Γ is the AMM quasi-symplectic groupoid.

Assume that X is a G-space. There is a natural map $i : H_2(X, \mathbb{Z}) \to H_2\big((G \times X)_\bullet, \mathbb{Z}\big)$ induced by the inclusion $C_2(X, \mathbb{Z}) \subset C_2((G \times X)_\bullet, \mathbb{Z})$. The following lemma indicates that i is in fact an isomorphism.

Lemma 5.8 *If G is a connected and simply-connected Lie group, then the map*

$$i : H_2(X, \mathbb{Z}) \to H_2((G \times X)_\bullet, \mathbb{Z})$$

is an isomorphism.

PROOF. This is a standard result. For completeness, we sketch a proof below. Let $G \to EG \to BG$ be the usual G-bundle over the classifying space BG and $X_G = G\backslash(EG \times X)$. We have the fibration $G \to EG \times X \to X_G$.

The second term of the homology Leray-Serre spectral sequence is $E^2_{p,q} = H_p(X_G, \mathcal{H}_q(G, \mathbb{Z}))$, *i.e.,* the homology of X_G with local coefficients in $\mathcal{H}_q(G, \mathbb{Z})$ (see [12]). Since G is simply-connected, we have $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$ and $E^2_{p,q}$ has the following form for $0 \le p \le 3$ and $0 \le q \le 2$:

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ H_0(X_G, \mathbb{Z}) & H_1(X_G, \mathbb{Z}) & H_2(X_G, \mathbb{Z}) & H_3(X_G, \mathbb{Z}) & \cdots \end{array} \tag{33}$$

According to Leray-Serre theorem, this spectral sequence converges to $H_*(EG \times X, \mathbb{Z})$. It is clear from Eq. (33) that, in particular, we have

$$H_2(EG \times X, \mathbb{Z}) \simeq H_2(X_G, \mathbb{Z}).$$

Since EG is contractible, we get

$$H_2(X_G, \mathbb{Z}) \simeq H_2(X, \mathbb{Z}).$$

The lemma now follows from the well-known isomorphism $H_2((G \times X)_\bullet, \mathbb{Z}) \simeq H_2(X_G, \mathbb{Z})$. □

Since $H_2(\mathfrak{g}^*, \mathbb{Z}) = 0$, Lemma 5.8 implies that

$$H_2\big((T^*G)_\bullet, \mathbb{Z}\big) = 0. \tag{34}$$

Since any simply-connected Lie group G satisfies $H_2(G, \mathbb{Z}) = 0$, we also have

$$H_2\big((G \times G)_\bullet, \mathbb{Z}\big) = 0. \tag{35}$$

Recall that the AMM quasi-symplectic groupoid is $(G \times G \rightrightarrows G, \omega + \Omega)$ [6, 22], where G is a compact Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form $(\cdot, \cdot)$. Here $G \times G \rightrightarrows G$ is the transformation groupoid, where G acts on itself by conjugation, and ω and Ω are defined as follows.

Following [3], we denote by θ and $\bar{\theta}$ the left and right Maurer-Cartan forms on G respectively, i.e., $\theta = g^{-1}dg$ and $\bar{\theta} = (dg)g^{-1}$. Let $\Omega \in \Omega^3(G)$ denote the bi-invariant 3-form on G corresponding to the Lie algebra 3-cocycle $\frac{1}{12}(\cdot, [\cdot, \cdot]) \in \wedge^3 \mathfrak{g}^*$ *i.e.*

$$\Omega = \frac{1}{12}(\theta, [\theta, \theta]) = \frac{1}{12}(\bar{\theta}, [\bar{\theta}, \bar{\theta}]), \tag{36}$$

and $\omega \in \Omega^2(G \times G)$ the 2-form

$$\omega|_{(g,x)} = -\frac{1}{2}[(Ad_x \operatorname{pr}_1^* \theta, \operatorname{pr}_1^* \theta) + (\operatorname{pr}_1^* \theta, \operatorname{pr}_2^*(\theta + \bar{\theta}))], \tag{37}$$

where (g, x) denotes the coordinate in $G \times G$, and pr_1 and $\operatorname{pr}_2 : G \times G \to G$ are natural projections. It is known that $\omega + \Omega$ is a integral 3-cocycle.

A triple (X, ω_X, J), where X is a manifold, ω_X is a G-invariant 2-form on X and $J : X \to G$ is a smooth map, is a quasi-Hamiltonian G-space in the sense of [3] if

(B1) the differential of ω_X is given by:

$$d\omega_X = J^*\Omega;$$

(B2) the map J satisfies

$$\hat{\xi} \lrcorner \omega_X = \frac{1}{2} J^*(\xi, \theta + \bar{\theta});$$

and

(B3) at each $x \in X$, the kernel of ω_X is given by

$$\ker \omega_X = \{\hat{\xi}(x) |\ \xi \in \ker(\operatorname{Ad}_{J(x)} + 1)\},$$

where $\hat{\xi}$ is the vector field on X associated to the infinitesimal action of $\xi \in \mathfrak{g}$ on X.

It is known [22] that these conditions are equivalent to $(X \stackrel{J}{\to} G, \omega_X)$ being a Hamiltonian Γ-space, where Γ is the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$. In this case, the integrality can be described in simpler terms as indicated in the following:

Proposition 5.9 *Let Γ be the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$, where G is a connected and simply-connected Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form. Let $(X \stackrel{J}{\to} G, \omega_X)$ be a quasi-Hamiltonian G-space. Assume that $\omega + \Omega$ is an integral 3-cocycle in $Z^3_{dR}((G \times G)_\bullet, \mathbb{Z})$. Then the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition if and only if $\forall C \in Z_2(X, \mathbb{Z})$ and $D \in C_3(G, \mathbb{Z})$ such that $dD = J_*(C)$,*

$$\int_C \omega_X - \int_D \Omega \ \in \ \mathbb{Z}. \tag{38}$$

Note that such D always exists for any $C \in Z_2(X, \mathbb{Z})$.

Proof. Note that we have the following commutating diagram of groupoid homomorphisms:

$$\begin{array}{ccc} X_\bullet & \stackrel{i}{\to} & (G \times X)_\bullet \\ \downarrow J & & \downarrow J \\ G_\bullet & \stackrel{J}{\to} & (G \times G)_\bullet, \end{array} \tag{39}$$

where $X_\bullet$ and $G_\bullet$ are spaces X and G are considered as groupoids, while $(G \times X)_\bullet$ and $(G \times G)_\bullet$ are the transformation groupoids. Thus one direction is obvious.

Conversely, according to Eq. (35), we have $H_2((G \times G)_\bullet, \mathbb{Z}) = 0$. Therefore

$$\mathrm{Ker}(J_*) = H_2\big((G \times M)_\bullet, \mathbb{Z}\big).$$

By Lemma 5.8, for any class $\mathcal{C} \in H_2\big((G \times X)_\bullet, \mathbb{Z}\big)$, there exists $C \in Z_2(X, \mathbb{Z})$ such that $\mathcal{C} = i_*[C]$. Since $H_2(G, \mathbb{Z}) = 0$, there always exists $D \in C_3(G, \mathbb{Z})$ such that $J_*(C) = dD$. Hence

$$J_*(i_* C) = i_*(J_* C) = i_* dD = \delta(i_* D).$$

Now it is clear that

$$\int_{i_* C} \omega_X - \int_{i_* D} (\omega + \Omega) = \int_C i^* \omega_X - \int_D i^*(\omega + \Omega) = \int_C \omega_X - \int_D \Omega.$$

The conclusion thus follows from Lemma 5.4. □

Remark 5.10 Note that Eq. (38) coincides with the quantization condition of Alekseev–Meinrenken [2]. See also [16]. For the case of conjugacy classes, see [14, 15].

As an immediate consequence of Proposition 5.9, we have the following:

Corollary 5.11 *Let Γ be the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$. Then $1 \in G$, considered as a quasi-Hamiltonian G-space, satisfies the integrality condition.*

Let us consider the case of Example 4.3 where Γ is the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$. In this case, we recover a well-known result of Guillemin–Sternberg [10].

Proposition 5.12 *Let Γ be the symplectic groupoid $(T^*G \rightrightarrows \mathfrak{g}^*, \omega)$, where G is a connected and simply-connected Lie group. Let $J : X \to \mathfrak{g}^*$ be a momentum map for a Hamiltonian G-space (X, ω_X) as in Example 4.3. The pair (ω_X, ω) satisfies the integrality condition if and only if ω_X is an integral 2-form.*

Proof. According to Eq. (34), we have $H_2((T^*G)_\bullet, \mathbb{Z}) = 0$. Therefore for any $C \in Z_2((G \times X)_\bullet, \mathbb{Z})$ there exists $D \in C_3((T^*G)_\bullet, , \mathbb{Z})$ such that $J_*(C) = \delta D$. By Lemma 5.8, we may assume that $C \in Z_2(X, \mathbb{Z})$. Since $H_2(\mathfrak{g}^*, \mathbb{Z}) = 0$, we can assume that $D \in C_3(\mathfrak{g}^*, \mathbb{Z})$. Since $\Omega = 0$, the integrality condition of Eq. (26) thus reads $\int_C \omega_X \in \mathbb{Z}$. □

In particular, a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, endowed with the Kirillov-Kostant-Souriau symplectic structure $\omega_{\mathcal{O}}$, satisfies the integrality condition if and only if $\omega_{\mathcal{O}}$ is an integral 2-form.

5.3 Integrality condition and Morita equivalence

In general, a pre-quasi-symplectic groupoid may not be exact, as in the case of the AMM-quasi-symplectic groupoid for instance. In such a case, one must pass to a Morita equivalent pre-quasi-symplectic groupoid in order to construct a prequantization. According to Theorem 4.12, Morita equivalent quasi-(pre)symplectic groupoids yield equivalent momentum map theories in the sense that there is a bijection between their (pre)-Hamiltonian Γ-spaces, and the classical intertwiner spaces are independent of Morita equivalence [22].

More precisely, given a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$, where Ω may not be exact, one can choose a surjective submersion $N \xrightarrow{p} M$ and consider the pull-back groupoid $\Gamma[N] \rightrightarrows N$ of $\Gamma \rightrightarrows M$ via p. Then $(\Gamma[N] \rightrightarrows N, p^*\omega + p^*\Omega)$ is again a pre-quasi-symplectic groupoid. Moreover, if $(X \xrightarrow{J} M, \omega_X)$ is a pre-Hamiltonian Γ-space, then $(X_N \xrightarrow{J_N} N, p^*\omega_X)$ is a pre-Hamiltonian $\Gamma[N]$-space, where $X_N = X \times_M N$, and $p : X_N \to X$ and $J_N : X_N \to N$ are the projections to the first and second components, respectively. The following proposition indicates that integrality condition is preserved under this pull-back procedure.

Lemma 5.13 *The pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition if and only if $\big(p^*\omega_X, p^*\omega + p^*\Omega\big)$ satisfies the integrality condition.*

Proof. By abuse of notation, we use the same letter p to denote the groupoid homomorphisms from $\Gamma[N] \times_N X_N \rightrightarrows X_N$ to $\Gamma \times M \rightrightarrows M$, and from $\Gamma[N] \rightrightarrows N$ to $\Gamma \rightrightarrows M$, both of which are Morita morphisms.

For any $C' \in Z_2((\Gamma[N] \times_N X_N)_\bullet, \mathbb{Z})$ and $D' \in C_3(\Gamma[N]_\bullet, \mathbb{Z})$ with $J_*(C') = \delta D'$, we have

$$\int_{C'} p^*\omega_X - \int_{D'} p^*(\omega + \Omega) = \int_{C} \omega_X - \int_{D} (\omega + \Omega), \tag{40}$$

where $C = p_*(C')$ and $D = p_*(D')$ clearly satisfy $J_*(C) = \delta D$.

Assume that the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition. Then Eq. (40) implies immediately that so too does the pair $(p^*\omega_X, p^*\omega + p^*\Omega)$.

Conversely, if $\big(p^*\omega_X, p^*\omega + p^*\Omega\big)$ satisfies the integrality condition, then $\omega + \Omega$ must be an integral cocycle. Now we have the commutative diagram:

$$\begin{array}{ccc} H_2((\Gamma[N] \times_N X_N)_\bullet, \mathbb{Z}) & \xrightarrow{p_*} & H_2((\Gamma \times_M X)_\bullet, \mathbb{Z}) \\ \downarrow J_{N*} & & \downarrow J_* \\ H_2(\Gamma[N]_\bullet, \mathbb{Z}) & \xrightarrow{p_*} & H_2(\Gamma_\bullet, \mathbb{Z}), \end{array}$$

where the horizontal arrows are isomorphisms. Therefore

$$p_* : H_2((\Gamma[N] \times_N X_N)_\bullet, \mathbb{Z}) \to H_2((\Gamma \times_M X)_\bullet, \mathbb{Z})$$

induces an isomorphism from $\mathrm{Ker}(J_{N*})$ to $\mathrm{Ker}(J_*)$. This implies that any class in $\mathrm{Ker}(J_*)$ has a representative of the form $C = p_*(C')$ where $C' = \delta D'$ for some $D' \in C_3((\Gamma[N] \times_N X_N)_\bullet, \mathbb{Z})$. Let $D = p_*(D')$. By Eq. (40), we see that if the pair $(p^*\omega_X, p^*\omega + p^*\Omega)$ satisfies the integrality condition then $\int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}$. By Lemma 5.4, we conclude that $(\omega_X, \omega + \Omega)$ satisfies the integrality condition. $\square$

Corollary 5.14 *Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ be Morita equivalent pre-quasi-symplectic groupoids. Assume that $(F \to G_0, \omega_F)$ and $(E \to H_0, \omega_E)$ are a pair of corresponding pre-Hamiltonian spaces. Then $(\omega_F, \omega_G + \Omega_G)$ satisfies the integrality condition if and only if $(\omega_E, \omega_H + \Omega_H)$ satisfies the integrality condition.*

PROOF. It suffices to prove this assertion for a Morita morphism of pre-quasi-symplectic groupoids. By Lemma 5.13, it remains to prove that the integrality condition is preserved by gauge transformations of the first type, which can be easily checked. $\square$

As a consequence, given a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$, where Ω may not be exact, one can choose a surjective submersion $N \xrightarrow{p} M$ such that $p^*\Omega \in \Omega^3(N)$ is exact and replace $(\Gamma \rightrightarrows M, \omega + \Omega)$ by a Morita equivalent exact pre-quasi-symplectic groupoid $(\Gamma[N] \rightrightarrows N, p^*\omega + p^*\Omega)$. Usually, one takes $N := \coprod U_i \to M$, where $\mathcal{U} = (U_i)$ is an open cover of M. Then the pull-back pre-quasi-symplectic groupoid is $(\Gamma[\mathcal{U}] \rightrightarrows \coprod U_i, \omega|_{\Gamma[\mathcal{U}]} + \Omega|_{U_i})$, where $\Gamma[\mathcal{U}]$, as a manifold, can be identified with the disjoint union $\coprod \Gamma_{U_i}^{U_j}$. Lemma 4.7 guarantees that the integrality condition always holds no matter which surjective submersion (or open covering) $N \to M$ is taken as long as the initial pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition, and therefore one can always construct a compatible prequantization.

Applying the above discussion to the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$ and using Theorem 5.7 groupoid, we are led to:

Corollary 5.15 *Let $(X \xrightarrow{J} G, \omega_X)$ be a quasi-Hamiltonian G-space. The following are equivalent*

1. *There exists a compatible prequantization $\coprod R_{ij} \to (G \times G)[\mathcal{U}] \rightrightarrows \coprod U_i$ and $\coprod L_i \to \coprod X|_{U_i}$, where $(G \times G)[\mathcal{U}] \rightrightarrows \coprod U_i$ is the pullback quasi-symplectic groupoid of the AMM groupoid using any open covering of G such that $\forall i$, $\Omega|_{U_i}$ is an exact form.*

2. *The integrality condition of Eq. (38) holds.*

5.4 Strong integrality condition

Definition 5.16 Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be a pre-quasi-symplectic groupoid. A pre-Hamiltonian Γ-space $(X \to M, \omega_X)$ is said to satisfy the *strong integrality condition* if

1. it satisfies the integrality condition; and
2. the map $J^* : H^2_{dR}(\Gamma_\bullet) \to H^2_{dR}((\Gamma \times_M X)_\bullet)$ vanishes.

The following result follows from Theorem 5.7.

Proposition 5.17 *Let $(\Gamma \rightrightarrows M, \omega+\Omega)$ be an exact proper pre-quasi-symplectic groupoid, and $(X \xrightarrow{J} M, \omega_X)$ a pre-Hamiltonian Γ-space. Then $(X \xrightarrow{J} M, \omega_X)$ satisfies the strong integrality condition if and only if for any prequantization of $(\Gamma \rightrightarrows M, \omega + \Omega)$, X admits a compatible prequantization.*

PROOF. If $(X \xrightarrow{J} M, \omega_X)$ satisfies the strong integrality condition, it is clear from Theorem 5.7 that X admits a compatible prequantization for any prequantization of $(\Gamma \rightrightarrows M, \omega + \Omega)$.

Conversely, given any prequantization $(R \to \Gamma \rightrightarrows M, \theta + B)$, $J^*(\theta + B) - \omega_X$ must be an integral 2-cocycle in $Z^2_{dR}((R \times_M X)_\bullet, \mathbb{Z})$. Note that if $\theta + B$ is a pseudo-connection, so is $\theta + B + \pi^*\Xi$, $\forall \Xi \in Z^2_{dR}(\Gamma_\bullet)$. Since the subset of integral classes $Z^2_{dR}(R \times_M X_\bullet, \mathbb{Z})$ is discrete, then $J^*(\theta + B) - \omega_X + J^*\pi^*\Xi$ being an integral cocycle for all Ξ implies that $[J^* \circ \pi^*(\Xi)] = 0$. In other words, the map $J^* \circ \pi^* : H^2_{dR}(\Gamma_\bullet) \to H^2_{dR}((R \times_M X)_\bullet)$ is the zero map. Since $R \times_M X \to \Gamma \times_M X \rightrightarrows X$ defines a trivial gerbe according to Proposition 2.3, the map $\pi^* : H^2_{dR}(\Gamma \times_M X) \to H^2_{dR}(R \times_M X)$ is injective. From the identity $J^* \circ \pi^* = \pi^* \circ J^*$ and the fact that π^* is injective, it follows that $J^* : H^2_{dR}(\Gamma_\bullet) \to H^2_{dR}((\Gamma \times_M X)_\bullet)$ must vanish. $\square$

The following proposition is an analogue of Corollary 5.14.

Proposition 5.18 *Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ be Morita equivalent pre-quasi-symplectic groupoids. Assume that $(F \to G_0, \omega_F)$ and $(E \to H_0, \omega_E)$ are a pair of corresponding pre-Hamiltonian spaces. Then $(\omega_F, \omega_G + \Omega_G)$ satisfies the strong integrality condition if and only if $(\omega_E, \omega_H + \Omega_H)$ satisfies the strong integrality condition.*

PROOF. By Corollary 5.14, we just have to check that Condition (2) in the definition of strong integrality condition is invariant under Morita equivalence. This follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} H^2_{dR}(G_\bullet) & \simeq & H^2_{dR}(H_\bullet) \\ \downarrow & & \downarrow \\ H^2_{dR}((G \times_{G_0} F)_\bullet) & \simeq & H^2_{dR}((H \times_{H_0} E)_\bullet), \end{array}$$

where the horizontal arrows are the natural isomorphism between the de Rham cohomologies of two Morita equivalent groupoids. $\square$

Remark 5.19 1. If the groupoid Γ satisfies $H^2_{dR}(\Gamma_\bullet, \mathbb{Z}) = 0$, then Condition (2) in the definition of the strong integrality is satisfied for any pre-Hamiltonian Γ-space. In this case, a pre-Hamiltonian space satisfies the integrality condition if and only if it satisfies the strong integrality condition.

2. If G is a connected and simply-connected Lie group, then $H_2((G \times G)_\bullet, \mathbb{Z}) = 0$. Therefore, any quasi-Hamiltonian G-space satisfying the integrality condition must satisfy the strong integrality condition.

The following proposition summarizes the results of this section.

Proposition 5.20 *Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an exact, proper, pre-quasi-symplectic groupoid, and $(X_k \stackrel{J_k}{\to} M, \omega_k)$, $k = 1, 2$, be pre-Hamiltonian Γ-spaces. Assume that $(X_1 \stackrel{J_1}{\to} M, \omega_1)$ satisfies the integrality condition while $(X_2 \stackrel{J_2}{\to} M, \omega_2)$ satisfies the strong integrality condition. Then there exists a prequantization of $(\Gamma \rightrightarrows M, \omega + \Omega)$ and compatible prequantizations of both X_1 and X_2. Therefore the classical intertwiner space $\overline{X_2} \times_\Gamma X_1$ is quantizable.*

Applying this result to the case of the AMM quasi-symplectic groupoid, we have the following

Corollary 5.21 *Let G be a connected and simply-connected compact Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form, and $(X \stackrel{J}{\to} G, \omega_X)$ a quasi-Hamiltonian G-space. Assume that ω_X satisfies the integrality condition as in Eq. (38). Then the reduced symplectic manifold $J^{-1}(1)/G$ is prequantizable, and the prequantization can be constructed using the prequantization of the AMM quasi-symplectic groupoid $(G \times G)[\mathcal{U}] \rightrightarrows \coprod U_i$ (more precisely the pull-back groupoid of the AMM quasi-symplectic groupoid) together with a compatible prequantization of the Hamiltonian space $(\coprod X|_{U_i} \to \coprod U_i, \omega_X|_{U_i})$, where $\mathcal{U} = (U_i)_{i \in I}$ is some open covering of G such that $\Omega|_{U_i}$ is exact $\forall i \in I$.*

6 Appendix

We denote then by C_{S^1} the canonical cycle in $C_1(S^1, \mathbb{Z})$ that generates $H_1(S^1, \mathbb{Z}) = \mathbb{Z}$. If we consider C_{S^1} as an element of $C_2(S^1_\bullet, \mathbb{Z})$, $[C_{S^1}]$ generates $H_2(S^1_\bullet, \mathbb{Z}) \simeq \mathbb{Z}$. For any point p in a manifold N, we denote by C_p the constant map from S^1 to $\{p\}$ and consider it as an element of $C_1(N, \mathbb{Z})$. Assume that $R \to \Gamma \rightrightarrows M$ is an S^1-central extension of groupoids. For any $m \in M$, let

$$Z_m = f_{m*}(C_{S^1}) \in C_2(R_\bullet, \mathbb{Z}), \tag{41}$$

where $f_m : S^1 \to R$ is defined by Eq. (12). More generally, for any $r \in R$, let $f_r : S^1 \to R$ be the map $\lambda \to \lambda \cdot r$, and set

$$Z_r = f_{r*}(C_{S^1}) - C_r \in C_2(R_\bullet, \mathbb{Z}).$$

Proposition 6.1 *Let $R \to \Gamma \rightrightarrows M$ be an S^1-central extension. Assume that M/Γ is connected.*

1. *The class $[Z_m] \in H_2(R_\bullet, \mathbb{Z})$ does not depend on the choice of $m \in M$. Because of this, we will drop the subscript m and denote this class simply by $[Z]$.*

2. *For any $r \in R$, Z_r is a cycle and $[Z_r] = [Z]$;*

3. *The natural map $\pi_* : H_2(R_\bullet, \mathbb{Z}) \to H_2(\Gamma_\bullet, \mathbb{Z})$ is surjective.*

4. *Its kernel Ker(π_*) is generated by $[Z]$.*

5. *The following identity holds:*
$$\int_{Z_m} \theta = 1.$$

Before we prove this proposition, we first need a lemma. Given any point $p \in N$, we will denote by $C_{p^{(k)}}$ the chain in $C_k(N, \mathbb{Z})$ defined by the constant path $\Delta_k \to \{p\}$.

Lemma 6.2 *Let $R \xrightarrow{\pi} \Gamma \rightrightarrows M$ be an S^1-central extension.*

1. *Any element E in $C_0(R, \mathbb{Z})$ with $\pi_*(E) = 0$ can be written of the form $E = \delta D'$, where $D' \in C_1(R, \mathbb{Z})$ satisfies $\pi_*(D') = 0$.*
2. *$\pi_* : C_\bullet(R_\bullet, \mathbb{Z}) \to C_\bullet(\Gamma_\bullet, \mathbb{Z})$ is a surjective map.*
3. *Any element in the kernel of $\pi_* : H_k(R_\bullet, \mathbb{Z}) \to H_k(\Gamma_\bullet, \mathbb{Z})$ has a representative $C \in Z_k(R_\bullet, \mathbb{Z})$ with $\pi_*(C) = 0$.*
4. *Any element C in $C_0(R_2, \mathbb{Z})$ with $\pi_*(C) = 0$ is of the form $C = dD'$, where $D' \in C_1(R_2, \mathbb{Z})$ satisfies $\pi_*(D') = 0$.*
5. *For any cycle $C' \in C_1(R, \mathbb{Z})$ such that $\delta C' = 0$ and $\pi_*(C') = 0$, we have*
$$[C'] = \sum_{i \in I} k_i [Z_{r_i}]$$
for some finite set I, $k_i \in \mathbb{Z}$ and $r_i \in R$.

Proof. 1) The kernel of $\pi_* : C_0(R, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z})$ is generated by elements of the form $p - q$, where p and q are two points in the same fibre of $R \xrightarrow{\pi} \Gamma$. Hence, it suffices to prove the claim for such a generator.

Let $D : \Delta_1 \to R$ be a path in the fiber $\pi^{-1}(p)$ satisfying $dD = p - q$. Set $D' = D - C_{p^{(1)}}$. Clearly, the identities $dD' = p - q$ and $\pi_*(D') = 0$ hold. Moreover, from $\pi_*(D') = 0$, it follows that $\partial D' = s_* \pi_*(D') - t_* \pi_*(D')$. Hence $\delta D' = p - q$

2) Since the projections $\pi : R_k \to \Gamma_k$ are surjective submersions with fibers isomorphic to k-dimensional torus, all the maps $\pi_* : C_l(R_k, \mathbb{Z}) \to C_l(\Gamma_k, \mathbb{Z})$ are onto for all $k, l \in \mathbb{N}$.

3) Let $C' \in Z_k(R_\bullet, \mathbb{Z})$ be a cycle with $\pi_*[C'] = 0$. By definition, there exists $D \in C_{k+1}(\Gamma_\bullet, \mathbb{Z})$ such that $\delta D = \pi_*(C')$. By 2), there exists D' in $C_{k+1}(R_\bullet, \mathbb{Z})$ such that $\pi_*(D') = D$. Set $C := C' - \delta D'$. We have $[C] = [C']$ and $\pi_*(C) = 0$.

4) The kernel of $\pi_* : C_0(R_2, \mathbb{Z}) \to C_0(\Gamma_2, \mathbb{Z})$ is generated by elements of the form $p - q$, where p and q are two points on the same fiber of $R_2 \to \Gamma_2$. It thus suffices to show this property for such generators.

Let $D : \Delta_1 \to R_2$ be a path in the fiber over $\pi(p)$ such that $dD = p - q$. Let $D' = D - C_{p^{(1)}}$. Thus
$$\pi_*(D') = \pi_*(D - C_{p^{(1)}}) = C_{\pi(p)^{(1)}} - C_{\pi(p)^{(1)}} = 0, \qquad \text{and} \quad dD' = p - q.$$

5) For simplicity, we call those chains in $C_1(R, \mathbb{Z})$ of the form $C - C_{p^{(1)}}$ *fibered* 1*-chains*, where $p \in R$ is a point and $C : \Delta_1 \to \pi^{-1}(\pi(p))$ is a path in the fiber through the point p. Any fibered 1-chain is in the kernel of π_* and hence lies in the kernel of ∂. If a fibered 1-chain E in a given

fiber satisfies $dE = 0$, then $[E] = k[Z_r]$ for some $r \in R$ and $k \in \mathbb{Z}$. As a consequence, if a linear combination of fibered 1-chains F is a cycle in $C_1(R, \mathbb{Z})$, then it is clear that $[F] = \sum_{i \in I} k_i [Z_{r_i}]$ for some finite set I, $k_i \in \mathbb{Z}$ and $r_i \in R$.

Now the kernel of $\pi_* : C_1(R, \mathbb{Z}) \to C_1(\Gamma, \mathbb{Z})$ is generated by elements of the form $C_0 - C_1$, where $C_i, i = 0, 1$, are paths $\Delta_1 \to R$ satisfying $\pi_*(C_0) = \pi_*(C_1)$. Thus there is a map $\gamma : \Delta_1 \to S^1$ such that $C_0(t) = \gamma(t) \cdot C_1(t)$ $\forall t \in \Delta_1$. Let $\widetilde{\gamma} : [0, 1] \times \Delta_1 \to S^1$ be a map with $\widetilde{\gamma}(0, t) = 1$ and $\widetilde{\gamma}(1, t) = \gamma(t)$. Let us define two maps D_1 and $\hat{D} : [0, 1] \times \Delta_1 \to R$ by $D_1(s, t) = \widetilde{\gamma}(s, t) \cdot C_0(t)$ and $\hat{D}(s, t) = C_0(t)$. Set $D := D_1 - \hat{D}$. We have $\pi_*(D) = 0$ and therefore $\partial D = 0$. Moreover, by construction, $C_0 - C_1 + \delta D = C_0 - C_1 + dD$ is the sum of two 1-fibered chains: one in the fiber through $C_0(0)$ and another in the fiber through $C_0(1)$. The conclusion thus follows. $\square$

Proof of Proposition 6.1. 1) and 2). It is clear that if m and n are in the same connected component of M, then $[Z_m] = [Z_n]$. Now by the definition of Z_r, we have

$$
\begin{aligned}
&dZ_r = d\big(f_{r*}(C_{S^1})\big) - dC_r = 0, \\
&s_*(Z_r) = s_*\big(f_{r*}(C_{S^1}) - C_r\big) = C_{s(r)} - C_{s(r)} = 0, \text{ and} \\
&t_*(Z_r) = t_*\big(f_{r*}(C_{S^1}) - C_r\big) = C_{t(r)} - C_{t(r)} = 0.
\end{aligned}
$$

Therefore $\delta(Z_r) = 0$. Consider the map $D : S^1 \to R_2$ defined by $\lambda \mapsto \big(f_r(\lambda), f_{t(r)}(\lambda^{-1})\big)$. We have $dD = 0$ and $\partial D = f_{r*}(C_{S^1}) - C_r - f_{t(r)*}(C_{S^1})$. Hence we have $[Z_r] = [Z_{t(r)}]$, $\forall r \in R$. Similarly, we have $[Z_r] = [Z_{s(r)}]$, $\forall r \in R$. Since M/Γ is connected, 1) and 2) follow.

3) Let $C \in Z_2(\Gamma_\bullet, \mathbb{Z})$ be any 2-cycle. According to Lemma 6.2 (2), there exists $D \in C_2(R_\bullet, \mathbb{Z})$ with $\pi_*(D) = C$.

In general, $\delta D \neq 0$. However since the restriction of π to M is the identity map, we have $\partial\ D_1\ -\ d\ D_2\ =\ 0$ and thus $\delta D = \partial D_0 - dD_1$, where $D = D_0 + D_1 + D_2,\ D_i \in C_i(R_{2-i}, \mathbb{Z})$. Therefore δD is an element of $C_0(R, \mathbb{Z})$ and $\pi_*(\delta D) = \delta \pi_*(D) = \delta C = 0$. By Lemma 6.2 (1), there exists $D' \in C_2(R_\bullet, \mathbb{Z})$ with $\pi_*(D') = 0$ and $\delta D' = \delta D$. Therefore it follows that $D - D'$ is a cycle in $Z_2(R_\bullet, \mathbb{Z})$ and

$$\pi_*([D - D']) = [\pi_*(D)] - [\pi_*(D')] = [C] - [0] = [C].$$

4) According to Lemma 6.2 (3), any class in $\mathrm{Ker}(\pi_*)$ has a representative C such that $\pi_*(C) = 0$ and therefore is of the form $C_0 + C_1$, where $C_0 \in C_0(R_2, \mathbb{Z})$ and $C_1 \in C_1(R, \mathbb{Z})$ satisfy $\pi_*(C_0) = 0$ and $\pi_*(C_1) = 0$. According to Lemma 6.2 (4), there exists $D' \in C_1(R_2, \mathbb{Z})$ with $\pi_*(D') = 0$ such that $C_0 = dD'$. Consider now $C' = C - \delta D' \in C_1(R, \mathbb{Z})$. We have

$$\delta C' = \delta C - \delta^2 D' = 0, \quad [C'] = [C], \quad \pi_*(C') = 0.$$

According to Lemma 6.2 (5), we have

$$[C'] = \sum_{i \in I} k_i [Z_{r_i}] \tag{42}$$

for some finite set I, $k_i \in \mathbb{Z}$ and $r_i \in R$. From Eq. (42), it follows that $[C] = [C'] = \sum_{i \in I} k_i [Z_{r_i}]$. By Lemma 6.1 (2), we have $[C] = (\sum_{i \in I} k_i)[Z]$.

5) holds because θ is a connection 1-form of the S^1-principal bundle $R \to \Gamma$. $\square$

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